

How to prove this polynomial always has integer values at all integers

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Abstract

The following problem was posed by user “Kevin” on Mathoverflow. How to prove this polynomial always has integer values at all integers?

$$P_m(x) = \sum_{i=0}^m \sum_{j=0}^m \binom{x+j}{j} \binom{x-1}{j} \binom{j}{i} \binom{m}{i} \binom{i}{m-j} \frac{3}{(2i-1)(2j+1)(2m-2i-1)}.$$

We provide an answer.

So

$$P_m(x) = \sum_{i=0}^m \sum_{j=0}^m \binom{x+j}{j} \binom{x-1}{j} \binom{j}{i} \binom{m}{i} \binom{i}{m-j} \frac{3}{(2i-1)(2j+1)(2m-2i-1)}.$$

Our task is to show it takes integer values on integers.

As Kevin explains at

[question 209140](<http://mathoverflow.net/q/209140>)

$P_m(x)$ is an even polynomial of degree $2m$ and he could show that $xP_m(x)$ always has integer values at all integers.

Folowing Wadim Zudilin we put

$$B_k(x) = \binom{x+k}{2k} + \binom{-x+k}{2k}.$$

For $k \geq 0$ the B_k are even polynomials of degree $2k$ that take integer values on integers. One has $B_k(k) = 1$ for $k \geq 1$, but $B_0(0) = 2$. Further $B_k(i) = 0$ for $|i| < k$. So the matrix

$$(B_k(i))_{0 \leq k \leq m}^{0 \leq i \leq m}$$

is triangular.

Every even polynomial $f(x)$ of degree $2j$ is clearly a linear combination of B_0, \dots, B_j and the coefficients are determined by $f(0), \dots, f(j)$. When $f(0) = 0$ it is actually a linear combination of B_1, \dots, B_j .

Rewrite $P_m(x)$ as

$$P_m(x) = \sum_k d(m, k) B_k(x)$$

with $d(m, k) \in \mathbb{Q}$. As explained by Kevin, $P_m(k)$ vanishes if $m > 2|k| - 2 \geq 0$ because all terms in the sum vanish. It can also be shown that $P_m(0) = 0$ for $m \geq 2$, but that is more tricky. Indeed we will show that $d(m, 0) = 0$ for $m \geq 2$.

Note that $P_m(x)$ visibly lies in the local ring $\mathbb{Z}_{(2)}$ for integer x . So it suffices to show that $d(m, k)$ lies in $\mathbb{Z}_{(p)}$ for any odd prime p . In fact we will find that the $d(m, k)$ are integers for $m \geq 1$. And $d(0, 0) = 3/2$ lies in $\mathbb{Z}_{(p)}$ for our odd prime p . For m not too large one may simply compute all $d(m, k)$. The matrix

$$(d(m, k))_{0 \leq m \leq 10}^{0 \leq k \leq 10}$$

looks like this

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 118 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 696 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 720 & 4824 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 336 & 8288 & 38240 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 60 & 6516 & 95928 & 336822 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2520 & 109872 & 1131732 & 3215544 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 392 & 67904 & 1735320 & 13647840 & 32651544 \end{pmatrix}.$$

We will tacitly use it to deal with small values of m .

We will study the set

$$V_p = \{(m, k) \in \mathbb{Z} \times \mathbb{Z} \mid d(m, k) \in \mathbb{Z}_{(p)}\}.$$

Using a method of Zeilberger we will prove relations between the $d(m, k)$ that were first discovered experimentally. One relation allows us to rewrite $m(m-1)(1+2m)d(m, k)$ in such a manner that we can use the method of Floors described in

[question 26336](<http://mathoverflow.net/q/26336>).

With that method we show that $m(m-1)(1+2m)d[m, k]$ is an integer multiple of $3m(m-1)$. Together with the relations this will allow us to show that V_p fills all of $\mathbb{Z} \times \mathbb{Z}$ for odd primes p .

Our variables i, j, k, m, n, q will take integer values only.

As in the A=B book [1] we use the convention that $\binom{x}{j}$ is a polynomial in x for fixed j . And it is the zero polynomial if $j < 0$. So $\binom{i}{j}$ is defined for all integers i, j . It also vanishes if $j > i \geq 0$. Of course $\binom{i}{j}$ agrees with the usual binomial coefficient if $0 \leq j \leq i$.

By inspecting the values at $x = 0, \dots, j$, we see that

$$(-1)^j \binom{x+j}{j} \binom{x-1}{j} - (-1)^{j-1} \binom{x+j-1}{j-1} \binom{x-1}{j-1}$$

equals $(-1)^j \binom{2j}{j} B_j(x)/2$ for $j \geq 0$. Taking the telescoping sum over j gives

$$(-1)^j \binom{x+j}{j} \binom{x-1}{j} = \sum_{k=0}^j (-1)^k \binom{2j}{j} B_k(x)/2$$

for $j \geq 0$. (Valid for all j , actually).

This allows us to conclude that

$$d(m, k) = \sum_{i=0}^m \sum_{j=k}^m \frac{3(-1)^{k+j} \binom{2k}{k} \binom{j}{i} \binom{m}{i} \binom{i}{m-j}}{2(2i-1)(2j+1)(2m-2i-1)}.$$

In particular $d(m, k) = 0$ for $m < 0$ and for $k > m$. We will see that $m(m-1)d(m, k)$ also vanishes for $2k-2 < m$.

Let us use the notation $[\text{statement}] = \begin{cases} 1, & \text{if statement is true;} \\ 0, & \text{otherwise.} \end{cases}$

Then

$$d(m, k) = \sum_{i,j} [j \geq k \geq 0] \text{term}(m, k, i, j), \quad (\Sigma ij) \tag{1}$$

where

$$\text{term}(m, k, i, j) = [m \geq 0] \frac{3(-1)^{k+j} \binom{2k}{k} \binom{j}{i} \binom{m}{i} \binom{i}{m-j}}{2(2i-1)(2j+1)(2m-2i-1)}.$$

Put

$$\begin{aligned}
\text{rel1}(m, k) = & \\
& -32(3-2k)^2(-k+m+1)(-k+m+2)d(m, k-2) \\
& +4(-k+m+1)(2km^2-2(k-1)(8k-9)m+(2k-3)(8(k-2)k+9))d(m, k-1) \\
& +k(-2k+m+2)(-2k+m+3)(-2k+2m+1)d(m, k),
\end{aligned}$$

$$\begin{aligned}
\text{rel2}(m, k) = & \\
& -4((m-1)^2-1)d(m-1, k-1) \\
& -4(2(k-1)+m+1)(-k+m+1)d(m, k-1) \\
& +k(2k-m-2)d(m, k)
\end{aligned}$$

Key results

- $\text{rel1}(m, k)$ vanishes.
- $m(m-1)d(m, k)$ vanishes for $2k-2 < m$.
- $\text{rel2}(m, k)$ vanishes.
- $m(m-1)(2m+1)d(m, k)$ is an integer multiple of $3m(m-1)$.

Before proving the Key results, let us draw conclusions from them. Let $m \geq 2$. As $d(m, 0) = 0$, we have $P_m(0) = 0$ and the $d(m, k)$ are determined by $P_m(1) \dots, P_m(m)$. Now the integral matrix

$$(B_k(i))_{1 \leq k \leq m}^{1 \leq i \leq m}$$

is triangular with ones on the diagonal. We conclude that $d(m, k) \in \mathbb{Z}_{(2)}$ for $m \geq 2$.

Let p be a prime, $p \geq 5$, and let $m \geq 2$. If p does not divide $2m+1$, then $d(m, k) \in \mathbb{Z}_{(p)}$ because $m(m-1)(2m+1)d(m, k) \in 3m(m-1)\mathbb{Z}_{(p)}$. Now assume p divides $2m+1$. Then it does not divide $2m+3$, so then $d(m+1, j) \in \mathbb{Z}_{(p)}$ for all j . Also, p does not divide $(m-1)(m+1)$, so it follows from $\text{rel2}(m+1, k+1) = 0$ that $d(m, k) \in \mathbb{Z}_{(p)}$. We have shown that $d(m, k) \in \mathbb{Z}_{(p)}$ if p is prime, $p \geq 5$, $m \geq 2$.

Remains $p = 3$. Let $m \geq 2$ again.

If 3 does not divide $2m+1$, then $d(m, k) \in 3\mathbb{Z}_{(3)}$ because $m(m-1)(2m+1)d(m, k) \in 3m(m-1)\mathbb{Z}_{(3)}$.

If $m \equiv 1 \pmod{9}$, or $m \equiv 7 \pmod{9}$, then $(2m+1)/3$ is prime to 3 and $d(m, k) \in \mathbb{Z}_{(3)}$ because $m(m-1)((2m+1)/3)d(m, k) \in m(m-1)\mathbb{Z}_{(3)}$.

If $m \equiv 4 \pmod{9}$, then $(m-1)(m+1)/3$ is prime to 3 and $d(m, k) \in \mathbb{Z}_{(3)}$ because $\text{rel}2(m+1, k+1) = 0$ shows $((m-1)(m+1)/3)d(m, k)$ is an integer linear combination of the integers $d(m+1, j)/3$.

We conclude that $d(m, k) \in \mathbb{Z}_{(3)}$ for $m \geq 2$. So the $d(m, k)$ are integers for $m \geq 2$ and P_m takes integer values on integers for $m \geq 2$. Recall that P_0, P_1 also take integer values. **Done.**

So we still have to prove the Key results.

First a technical issue. If $x > 0$ then $\binom{x}{j} = \frac{\Gamma(1+x)}{\Gamma(1+j)\Gamma(1+x-j)}$ and the bimeromorphic function

$$f(x, y) = \frac{\Gamma(1+x)}{\Gamma(1+y)\Gamma(1+x-y)}$$

is continuous at (x, j) . However, if $i < 0$ then f has an indeterminate value at (i, j) . For example, $\binom{i}{i}$ equals 1 if $i \geq 0$, but it vanishes for $i < 0$. At $(-1, -1)$ both 0 and 1 are values of f . Indeed Mathematica can be steered to give either answer.

`Binomial[i, j] /. i -> -1 /. j -> -1` gives 1 and

`Binomial[i, j] /. j -> -1 /. i -> -1` gives 0.

And `FullSimplify[Binomial[i, i] == Binomial[i - 1, i - 1]]` yields **True**. This answer is correct, but it tells only that for generic complex numbers i the identity holds.

Thus we need to make case distinctions when using identities between multimeromorphic functions, explicitly or implicitly, to prove identities involving the $\binom{i}{j}$.

We start proving that $\text{rel}1(m, k)$ vanishes.

As $[j \geq k+1](2(2k+1)\text{term}(m, k, i, j) + (k+1)\text{term}(m, k+1, i, j)) = 0$, we get from (Σij) that

$$2(2k+1)d(m, k) + (k+1)d(m, k+1) = \sum_i \text{iterm}(m, k, i) \quad (\Sigma i)$$

where

$$\text{iterm}(m, k, i) = 2(2k+1)\text{term}(m, k, i, k).$$

Now we use the

Fast Zeilberger Package version 3.61

written by Peter Paule, Markus Schorn, and Axel Riese

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Johannes Kepler University, Linz, Austria.

It suggests to put

$$g(m, k, i) = \frac{3 \times 2^{2k+3} m(-2i + m + 1) \Gamma\left(k + \frac{3}{2}\right) \binom{k+1}{i-1} \binom{m-1}{k+1} \binom{k+1}{m-i}}{\Gamma\left(\frac{1}{2}\right) \Gamma(k+2)}$$

and show that

$$\begin{aligned} & -32(1+2k)(3+2k)(k-m)(1+k-m) \text{iterm}(m, k, i) \\ & -4(1+k-m)(57+110k+72k^2+16k^3-34m-46km-16k^2m+4m^2+2km^2) \\ & \quad \times \text{iterm}(m, k+1, i) \\ & -(2+k)(5+2k-2m)(3+2k-m)(4+2k-m) \text{iterm}(m, k+2, i) \\ & -g(m, k, i+1) + g(m, k, i) = 0 \end{aligned}$$

for $m \geq 0$. So we do that and then sum over i , using (Σi) . The g terms drop out by telescoping and we get a relation

$$\begin{aligned} & -32(1+2k)(3+2k)(k-m)(1+k-m)(2(2k+1)d(m, k) + (k+1)d(m, k+1)) \\ & -4(1+k-m)(57+110k+72k^2+16k^3-34m-46km-16k^2m+4m^2+2km^2) \\ & \quad \times (2(2k+3)d(m, k+1) + (k+2)d(m, k+2)) \\ & -(2+k)(5+2k-2m)(3+2k-m)(4+2k-m) \\ & \quad \times (2(2k+5)d(m, k+2) + (k+3)d(m, k+3)) \\ & = 0 \end{aligned}$$

valid for all m , as it is obvious for $m < 0$. We may rewrite it as a recursion for rel1 :

$$2(3+2k)\text{rel1}(m, k+2) + (2+k) \text{rel1}(m, k+3) = 0.$$

As $d(m, k)$ vanishes for $k > m$, it follows from the recursion that $\text{rel1}(m, k)$ vanishes for all k .

So we have established the vanishing of $\text{rel1}(m, k)$.

Put

$$\text{pterm}(m, x, i, j) = \binom{x+j}{j} \binom{x-1}{j} \binom{j}{i} \binom{m}{i} \binom{i}{m-j} \frac{3}{(2i-1)(2j+1)(2m-2i-1)},$$

so that

$$P_m(x) = \sum_{i,j} \text{pterm}(m, x, i, j).$$

If $k \geq 1$ and $\text{pterm}[m, k, i, j]$ is nonzero, then $k - 1 \geq j$ and $m \geq j \geq i \geq m - j$. We see that

$$P_m(k) = 0 \text{ if } 0 \leq 2k - 2 < m,$$

because all the $\text{pterm}(m, k, i, j)$ vanish. In particular we get

$$0 = P_m(1) = \sum_k d(m, k) B_k(1) = 2d(m, 0) + d(m, 1),$$

and

$$0 = P_m(2) = \sum_k d(m, k) B_k(2) = 2d(m, 0) + 4d(m, 1) + d(m, 2)$$

for $m \geq 3$. So then $d(m, 1) = -2d(m, 0)$ and $d(m, 2) = 6d(m, 0)$. Substitute this into $\text{rel1}(m, 2) = 0$ and you find

$$4m(m - 1)(-2m - 1)d(m, 0) = 0.$$

This means that $d(m, 0) = 0$ for $m \geq 3$. As $d(2, 0)$ also vanishes, we now know that $m(m - 1)P_m(k)$ vanishes if $m > 2|k| - 2$. As the matrix

$$(B_k(i))_{0 \leq k \leq m}^{0 \leq i \leq m}$$

is triangular, we now conclude that

$$m(m - 1)d(m, k) \text{ vanishes for } m > 2k - 2. \quad (\text{SSE})$$

So we have established the vanishing of $m(m - 1)d(m, k)$ for $m > 2k - 2$.

Before turning to $\text{rel2}(m, k)$ we compute $d(2k - 2, k)$ and $d(2k - 3, k)$ for $k \geq 3$. These are the values that help to compute all $d(m, k)$ recursively with the recursion given by $\text{rel1}(m, k) = 0$. As $d(2k - 2, j)$ vanishes for $j < k$, one has

$$d(2k - 2, k) = P_{2k-2}(k) = \text{pterm}(2k - 2, k, k - 1, k - 1)$$

and similarly

$$d(2k - 3, k) = P_{2k-3}(k) = \text{pterm}(2k - 3, k, k - 1, k - 2) + \text{pterm}(2k - 3, k, k - 1, k - 1).$$

So we know $d(m, k)$ for $m \geq 2k - 3 \geq 3$. By (SSE) we also know $d(m, k)$ for $k \leq 1$ and any m . Using these values we get $\text{rel2}(m, k) = 0$ by inspection for $m \geq 2k - 3$ or $k \leq 1$. Notice that $(-7 + 2k)\text{rel2}(2k - 4, k) - \text{rel1}(2k - 4, k)$ is a combination of the known terms $d(-5 + 2k, -1 + k)$, $d(-4 + 2k, -2 + k)$, $d(-4 + 2k, -1 + k)$. It also vanishes by inspection, so we now have that $\text{rel2}(m, k) = 0$ for $m \geq 2k - 4$ or $k \leq 1$.

By substituting the definitions and expanding we check that

$$\begin{aligned}
& (-1+k)(-1+2k-2m)(-3+2k-m)(4-2k+m)\text{rel2}(m,k) \\
& + 4(-1+(-1+m)^2)\text{rel1}(m-1,k-1) \\
& - 32(5-2k)^2(-2+k-m)(-1+k-m)\text{rel2}(m,k-2) \\
& + 4(1-k+m) \\
& \times (-99+16k^3-2m(32+m)-8k^2(11+2m)+2k(81+m(31+m)))\text{rel2}(m,k-1) \\
& - (-1+k)(-4+2k-m)\text{rel1}(m,k) \\
& - 4(-1+k-m)(-5+2k+m)\text{rel1}(m,k-1) \\
& = 0
\end{aligned}$$

As rel1 vanishes, this leads to the following recursion for rel2 .

$$\begin{aligned}
& (-1+k)(-1+2k-2m)(-3+2k-m)(4-2k+m)\text{rel2}(m,k) \\
& - 32(5-2k)^2(-2+k-m)(-1+k-m)\text{rel2}(m,k-2) \\
& + 4(1-k+m) \\
& \times (-99+16k^3-2m(32+m)-8k^2(11+2m)+2k(81+m(31+m)))\text{rel2}(m,k-1) \\
& = 0
\end{aligned}$$

As $\text{rel2}(m,k) = 0$ for $2k-4 \leq m$ or $k \leq 1$, the recursion shows by induction on k that $\text{rel2}(m,k) = 0$ for all m, k .

So we have also established the vanishing of $\text{rel2}(m,k)$ and it is time to show the Key result that $m(m-1)(2m+1)d(m,k)$ is an integer multiple of $3m(m-1)$. This is obvious for $m < 2$, so we further assume $m \geq 2$. Then we know that $d(m,0) = 0$ and we have seen this implies $d(m,k) \in \mathbb{Z}_{(2)}$. So it suffices to show that $m(m-1)(2m+1)d(m,k) \in 3m(m-1)\mathbb{Z}[1/2]$.

Using relation (Σi) we may rewrite $\text{rel1}(m,k) = 0$ as

$$\begin{aligned}
& 2(m-1)m(2m+1)d(m,k-1) \\
& + (2-2k+m)(3-2k+m)(1-2k+2m) \sum_i \text{iterm}(m,k-1,i) \\
& + 16(3-2k)(-2+k-m)(-1+k-m) \sum_i \text{iterm}(m,k-2,i) \\
& = 0
\end{aligned}$$

We claim that

$$(2 - 2k + m)(3 - 2k + m)(1 - 2k + 2m)\text{iterm}(m, k - 1, i) \\ + 16(3 - 2k)(-2 + k - m)(-1 + k - m)\text{iterm}(m, k - 2, i)$$

lies in $3m(m - 1)\mathbb{Z}[1/2]$.

That will prove that the $(m - 1)m(2m + 1)d(m, k - 1)$ are integer multiples of $3m(m - 1)$.

Put

$$\text{frac1}(m, k, i) = \frac{3(m - 1)m \binom{2(k-1)}{k-1} (-2k + 2m + 1) \binom{k-1}{i} \binom{m}{i} \binom{i}{-k+m+1}}{(2i - 1)(2m - 2i - 1)}$$

and

$$\text{frac2}(m, k, i) = 6(k - m - 1) \binom{2(k-1)}{k-1} \binom{k-1}{i} \binom{m}{i} \binom{i}{-k+m+1}.$$

Then $\text{frac1}(m, k, i) + \text{frac2}(m, k, i)$ equals

$$(2 - 2k + m)(3 - 2k + m)(1 - 2k + 2m)\text{iterm}(m, k - 1, i) \\ + 16(3 - 2k)(-2 + k - m)(-1 + k - m)\text{iterm}(m, k - 2, i),$$

so it suffices to show that $\text{frac1}(m, k, i)/(6m(m - 1))$ and $\text{frac2}(m, k, i)/(6m(m - 1))$, which make sense for $m \geq 2$, lie in $\mathbb{Z}[1/2]$ for $m \geq 2$. Recall that the Catalan numbers

$$C(i) = \frac{\binom{2i}{i}}{i + 1}$$

are integers. See

[A000108](<https://oeis.org/A000108>)

We now look at $\text{frac1}(m, k, i)/(6m(m - 1))$.

If $\text{frac1}(m, k, i)$ is nonzero then $m \geq k - 1 \geq i \geq m + 1 - k \geq 0$. We distinguish two cases: $m = k - 1 \geq i \geq 0$ and $m > k - 1 \geq i \geq m + 1 - k \geq 0$.

First let $m = k - 1 \geq i \geq 0$. If $i = k - 1$, then

$$\text{frac1}(m, k, i)/(6m(m - 1)) = \text{frac1}(k - 1, k, k - 1)/(6(k - 1)(k - 2)) = C(k - 2).$$

Similarly $\text{frac1}(k - 1, k, 0)/(6(k - 1)(k - 2)) = C(k - 2)$.

So we may assume $0 < i < m = k - 1$. Then

$\text{frac1}(m, k, i)/(6m(m - 1)) = \text{frac1}(m, m + 1, i)/(6m(m - 1))$ equals

$$\frac{-(2i-2)!(2m)!(-2i+2m-2)!}{2(i!)^2(2i-1)!((m-i)!)^2(-2i+2m-1)!}$$

and we must show it takes values in $\mathbb{Z}[1/2]$.

This is the kind of expression to which one may apply the method of Floors explained in [question 26336](<http://mathoverflow.net/q/26336>).

It is based on

$$\text{ord}_p n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

According to the method it suffices to check that $\text{test}(m, i, 2n+1) \geq 0$ for $n \geq 1$, where

$$\begin{aligned} \text{test}(m, i, q) = & -2 \left\lfloor \frac{m-i}{q} \right\rfloor + \left\lfloor \frac{-2i+2m-2}{q} \right\rfloor - \left\lfloor \frac{-2i+2m-1}{q} \right\rfloor \\ & -2 \left\lfloor \frac{i}{q} \right\rfloor + \left\lfloor \frac{2i-2}{q} \right\rfloor - \left\lfloor \frac{2i-1}{q} \right\rfloor + \left\lfloor \frac{2m}{q} \right\rfloor. \end{aligned}$$

This is a tedious puzzle. For fixed q the function $\text{test}(m, i, q)$ is periodic of period q in both variables i and m . So for fixed q one may simply compute all values. We do it for $3 \leq q = 2n+1 < 17$. The results are nonnegative. But if q is large we need to be more efficient. If both $q = 2n+1$ and m are fixed, then $\text{test}(m, i, q)$ can only change value where at least one of the Floors jumps as a function of i . So it suffices to sample around the jumping points (modulo q). We know where they are. More specifically, we only need to consider the 15 cases where one of $i-1, i, i+1$ lies in $\{0, 1, -1+m, m, -2+m-n, -1+m-n, 1+n\}$. So we can eliminate i at the expense of having 15 cases. Similarly we can eliminate m for each of those cases, ending up with 153 test functions that depend on n only. Each test function is a linear combination of seven Floors. Each of the Floors stabilises after n has reached an easily computable bound. For instance $\left\lfloor -\frac{8}{2n+1} \right\rfloor$ is constant for $n \geq 4$. In fact the bound 5 suffices for all 7×153 Floors. Compute the 153 stable values. They are nonnegative. This solves the puzzle; the check for $3 \leq q = 2n+1 < 17$ was overkill.

So we now turn to the case $m > k-1 \geq i \geq m+1-k \geq 0$. Then

$$\begin{aligned} \text{frac1}(m, k, i)/(6m(m-1)C(i-1)) = & \frac{i!(2k-2)!m!(-2i+2m-2)!(-2k+2m+1)!}{(2i)!(k-1)!(-i+k-1)!(m-i)!(-2i+2m-1)!(-k+m+1)!(2m-2k)!(i+k-m-1)!} \end{aligned}$$

We use the method of Floors again to show that $\text{frac1}(m, k, i)/(6m(m-1)C(i-1)) \in \mathbb{Z}[1/2]$. This time we eliminate k, m, i in that order and take $n \geq 6$ as bound where all 13×3508 Floors are stable.

So we have shown that $\text{frac1}(m, k, i)/(6m(m-1))$ lies in $\mathbb{Z}[1/2]$ for $m \geq 2$. Remains showing that $\text{frac2}(m, k, i)/(6m(m-1))$ lies in $\mathbb{Z}[1/2]$ for $m \geq 2$.

If $\text{frac2}(m, k, i)$ is nonzero then $m > k-1 \geq i \geq m+1-k > 0$ and $\text{frac2}(m, k, i)/(6m(m-1))$ equals

$$\frac{-(2k-2)!(m-2)!}{i!(k-1)!(-i+k-1)!(m-i)!(m-k)!(i+k-m-1)!}.$$

This can be treated like the previous case. We eliminate k, m, i in that order and take $n \geq 6$ as bound where all 8×1278 Floors are stable.

Done

References

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